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On Two Monodromy Problems for Curves in Positive Characteristic

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The theory of limit linear series developed a few years ago by D. Eisenbud and J. Harris in [EH1] (and applied by them in several papers, e.g., [EH2, EH3]) is one of the fundamental tools to prove results on algebraic curves (in characteristic zero). The full power of this theory is not available in positive charateristic. Hence to prove in positive characteristic the corresponding results it is very natural to use "only" the fact that the result is true in characteristic zero. Of course, the reduction to the characteristic zero case can be nontrivial or depend on deep results. A common tool for the reduction for the two problems considered here is Enriques–Zariski connectedness theorem [EGA, III 4.3] (which was used exactly for the same purposes in [DM, F]). Here we are interested in monodromy problems: prove that a certain Galois group which acts on a finite set of objects defined over a "generic" curve is "big" (e.g., that it is the full symmetric group on these objects). One way to measure the bigness of a permutation group is the fact that it is at least k-transitive for suitable k. The k-transitivity for any given $k>0$ has a geometric translation in terms of irreducibility of a certain fiber product iterated k times (if the Galois group is the Galois group of the normal extension of the function fields of a finite separable morphism $f: A \to B$ with A reduced and B integral, then k-transitivity means the irreducibility of the total space of the fiber product of f with itself k times) (see [BH, R, B] for a discussion tailor made for the positive characteristic case). The first result of this note is concerned with the monodromy group of the "linear series of the general curve of genus g when the Brill–Noether number ρ vanishes" (see [EH2] for the meaning of these words). More precisely, we prove the following result.

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THEOREM 0.1. *Fix integers g, d, r with $g \geqslant 3$, $r > 0$, $d > 0$, and Brill–Noether number $\rho(g, d, r) := g - (r+1)(g+r-d) = 0$. Fix a prime $p > 0$ and algebraically closed fields* $\mathbf{K}$ *and* $\mathbf{L}$ *with* $\operatorname{char}(\mathbf{K}) = p$ *and* $\operatorname{char}(\mathbf{L}) = 0$. *Assume that the monodromy group of the g^r_d's on a general curve of genus g over* $\mathbf{L}$ *is k-transitive. Then the monodromy group of the g^r_d's on a general curve of genus g over* $\mathbf{K}$ *is k-transitive.*

In [EH2] it is proved in characteristic zero that the corresponding Galois group contains many elements (see also the discussion in [BC, problem 1]); in particular this group is always transitive and it is the full symmetric group if $r = 1$.

We remark that knowing that a finite group G acting on a set with t elements is k-transitive for some $k \geqslant 3$ gives a lot of informations by the existence of a very short list of finite permutation groups which are at least 3-transitive (see [R, Thm. 2.4]; note the obvious slip and add to that list M_{22} among the 3-transitive groups). In particular if, say, $k > 5$ then G is either the full symmetric group S_t or the alternating group $A(t)$.

A simular but easier problem is the one about the monodromy of Weierstrass points (on the "generic" curve) [EH3]. There are two main differences. Now it is always true (and proven in [EH3]) that in characteristic zero the monodromy group is the full symmetric group (the good difference). The bad difference is that the Hurwitz scheme $H_{n,w}$ (see [F]) is not "finite over $\mathbf{Z}$ but only over all primes $p > n$" (see [F, Thm. 7.2] for the meaning of the words between quotation marks) (this, as explained in [F], was the reason why in [F] it was proved the irreducibility of the moduli scheme of genus g curves only in large characteristic, say for $p > g+1$). Putting together these two differences, one obtains for free in the same way (details left to the reader) the following result.

THEOREM 0.2. *Fix an integer $g \geqslant 2$ and a prime $p > g+1$. Then the monodromy group of the Weierstrass points on the generic genus g smooth curve is the full symmetric group.*

Remark 0.3. In the same way the reader can translate the part related to the k-transitivity of [EEHS, Sect. 4] in characteristic p if $p > g+1$.

1. PROOF OF THEOREM 0.1

Fix p, g, d, r as in the statement of Theorem 0.1. It is known the existence of an open subset U of $\operatorname{Spec}(\mathbf{Z})$ with $(p) \in U$, and of a smooth open subset M' of the moduli scheme of genus g connected smooth curves M_g (over U) which intersects every fiber over U (for the smoothness at each prime see

[L] (or use [DM, Thm. 1.6] at the level of formal deformations)) on which there exists a universal curve $C' \to M'$. Taking d times the fiber product of $C' \to M'$ with itself, we find a smooth scheme $f\colon T \to U$ with the following properties:

(i) there is a universal curve $\pi\colon C'' \to T$ with d ordered sections $s_1, ..., s_d$;

(ii) for every prime $r \in U$, the generic curve over the algebraic closure of the field F_r is isomorphic to a fiber of π.

The main point is that the morphism $T \to M'$ is (proper, smooth) with geometrically integral fibers. Since π has a section, there is the relative Picard scheme $P := \mathrm{Pic}^0_\pi$ of π; call $\tau\colon P \to T$ the natural proper morphism. By using the section s_1 we may twist the tautological degree 0 line bundle on $P \times_\pi C$ and obtain a tautological degree d line bundle, L, on $P \times_\pi C$. Denote by h the projection $P \times_\pi C \to P$. Set $\Pi' := \{x \in P : h^0(L \mid h^{-1}(x)) \geqslant r\}$ and $\Pi'' := \{x \in P : h^0(L \mid h^{-1}(x)) > r\}$. By Brill–Noether theory and the fact that $\rho(d, g, r) \geqslant 0$, we have $T = \tau(\Pi')$. Set $V := T \setminus \tau(\prod'')$. Since Brill–Noether conjecture and Petri's conjecture hold in positive characteristic (see the original proof of Gieseker–Petri theorem/conjecture in the fundamental paper [Gi]), the open set V intersects each fiber of the morphism $T \to U \subseteq \mathrm{Spec}(\mathbf{Z})$. Thus on V it is defined "the scheme of g^r_d's" (using the standard determinantal approach given, e.g., in [ACGH, Chap. 4], and, of course, the existence of the d sections s_i); call it G. We do not need (and do not claim) the existence of the corresponding scheme of all g^r_d's on M'; it exists over the algebraic closure of each point of $\mathrm{Spec}(\mathbf{Z})$ and we only need that the scheme on which it lives maps with smooth and integral fibers to M'. Hence one can compute for $\mathbf{K}$ and for $\mathbf{L}$ the corresponding monodromy group upstairs on V using G. Almost by definition the k-transitivity of the monodromy group is equivalent to the irreducibility of the iteration k times of the fiber product of the morphism $q\colon G \to V$. We claim that the morphism q is smooth (hence G is smooth, because T is smooth). Assume the claim. Note that the corresponding iterated k-times fiber product of q is again a smooth and proper morphism. Hence Theorem 0.1 follows from Enriques–Zariski connectedness theorem ([EGA, III 4.3]; here are the essential points: a quasi-finite morphism can be compactified to a finite morphism, a smooth quasi-finite morphism with fibers of constant cardinality is finite, and then apply Coro. 4.3.2 and Remarque 4.3.4 of [EGA, III 4.3]). Now we prove the claim. By Gieseker–Petri theorem (in any characteristic [Gi]) and its interpretation (and the assumption on $\rho(g, d, r)$), q is finite and has smooth fibers with constant cardinality (again after a shrinking of the base). Hence it is sufficient to check the flatness of q. Since T is smooth and q is finite, the flatness of q

is equivalent to the fact that G is Cohen–Macaulay. By Brill–Noether construction (here we use the existence of the d sections) G is a determinantal subscheme of the expected codimension of a smooth Noetherian scheme. Hence G is Cohen–Macaulay by [HE, Coro. 4].

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